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PROBLEM OF STABILITY OF ELASTOPLASTIC EQUILIBRIUM OF A CENTRALLY COMPRESSED ROD OF CRUCIFORM TRANSVERSE CROSS SECTION

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Iu. A. CHERNIAKOV and N. Iu. SHVAIKO
(Dnepropetrovsk)
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The problem of bifurcation of the modes of elastoplastic equilibrium of a centrally compressed rod of cruciform transverse cross section is solved. Use is made of the nonlinear differential relations connecting the stress and strain variations at instability, obtained within the framework of the model of a linear, anisotropically self-hardening, plane medium [1] and the isotropy postulate of Il'iushin [2]. The solution is compared with the already known results [3] following from the deformation theory and the incremental theory of plasticity. It is shown that the above theories cannot be used in solving the problem under consideration, whether it is posed in the Kármán's, or in the Shanley's formulation.

We know that the instability in thin-walled elements is accompanied, as a rule, by a break in the load trajectory, the angle of break being of arbitrary magnitude, i.e. the process differs appreciably from the case of simple loading. In the vicinity of the break point the relationship between the stress and strain increments depends significantly on the angle of break, consequently this relationship must be expressed (in contrast to the deformation and incremental theories), by nonlinear differential relations. As far as the authors are aware, all attempts made to apply such relations obtained in certain versions of the theory of plasticity (see [4, 5] et al.) to solving the problems of stability, have encountered considerable difficulties of mathematical nature and did not lead to any positive results.

Below we make such an attempt in the course of solving a problem of determining the point of bifurcation of the modes of equilibrium of a centrally compressed rod of cruciform transverse cross section in both, the Kármán's and the Shanley's formulation.

1. On the relationships connecting the stress and strain increments in the problems on stability. For the model of a linear, anisotropically self-hardening plane medium [1] the relationship between the stress and strain increments $\delta\sigma$ and $\delta\varepsilon$, respectively, in a close neighborhood of the break point on the load trajectory was established in [6, 7]. A generalization to the three-dimensional case was achieved using the isotropy postulate. Three regions of additional loading were shown, in which the relation connecting the increments $\delta\sigma$ and $\delta\varepsilon$ is different (see Fig. 1 which corresponds to the case, discussed below, of a bending-torsional mode of instability of a compressed rod). In zone I (angle of break $\beta \leqslant \beta_0(\sigma_0)$) the relation can be obtained from the deformation theory of plasticity. An explicit expression for the relation $\delta\sigma \sim \delta\varepsilon$ and, in particular, the formulas for determining the functions $\beta_0(\sigma_0)$ and

$\beta_* (\sigma_0)$, are found in [6, 7]. In zone II ($\beta_0 \leqslant \beta \leqslant \beta_* (\sigma_0)$) the relation $\delta\sigma \sim \delta\varepsilon$ is nonlinearly differential and conforms neither to the theory of small elastoplastic deformations, nor to the incremental theory of plasticity, while in zone III when $\beta \in [\beta_* (\sigma_0), \pi]$, unloading takes place according to an elastic law.

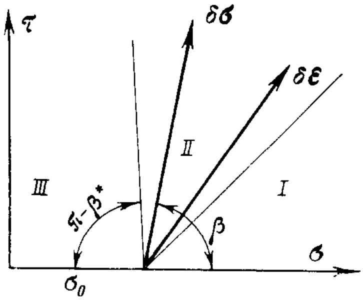

Fig. 1

The fact that the analytical expressions for the relation $\delta\sigma \sim \delta\varepsilon$ are different for each of the regions of additional loading shown above and the awkwardness of the relations obtained in [6, 7], made them inconvenient for the straightforward application in solving the problems of stability. For this reason the authors of [8, 9] derived approximate expressions for the defining equations near the break point, using the same type of analytic expressions for all three regions. Below we give these approximating expressions in their final form, written here for the case of bending-torsional instability of the rod

$$\delta\sigma_z = \frac{3}{2}k\left[a_1\delta\varepsilon_z - \left(\frac{a_0}{2} + a_2\right)\delta\varepsilon_u + \frac{2a_2\delta\gamma^2}{3\delta\varepsilon_u}\right] \tag{1.1}$$

$$\delta\tau = \frac{1}{2k}\left[b_1 - 2b_2\frac{\delta\varepsilon_z}{\delta\varepsilon_u}\right]\delta\gamma \qquad \left(\delta\varepsilon_u = \left(\delta\varepsilon_z{}^2 + \frac{1}{3}\delta\gamma^2\right)^{1/2}\right)$$

Here the coefficients a_i and b_i ($i = 0, 1, 2$) depend on the material and the magnitude of the stress σ_0 at the break point (Fig. 1). In particular, for the aluminium alloy АМГ (*) it was obtained

$$\begin{gathered}
{}^1/_2\, a_0 = 23.98\, \alpha_0{}^2 - 22.78\, \alpha_0 - 7.19, \quad a_1 = 55.95\, \alpha_0{}^2 - 26.87\, \alpha_0 + 18.04 \\
a_2 = 22.11\, \alpha_0{}^2 + 6.42\, \alpha_0 - 5.06, \quad b_1 = -59.69\, \alpha_0{}^2 - 13.86\, \alpha_0 + 34.09 \\
b_2 = -6.63\, \alpha_0^2 - 21.27\, \alpha_0 + 0.23, \quad G = 0.273 \cdot 10^6 \text{ kg/cm}^2 \\
\tau_s = 325 \text{ kg/cm}^2, G / k = 17, \quad \sigma_s = \sqrt{3}\,\tau_s
\end{gathered} \tag{1.2}$$

Values of the coefficients a_i and b_i for the aluminium alloys Al-10 and Д-16 are given in [9].

The relation between the parameter α_0 and the quantity σ_0 / σ_s is given by

$$I_0 (2\, \alpha_0) - 2\, \alpha_0\, I_1 (2\, \alpha_0) \ln (0.21 / \alpha_0) = \sigma_s / \sigma_0 \tag{1.3}$$

We note that the indicated version of the theory was confirmed experimentally, within the range of small elastoplastic deformations, when aluminium alloys were loaded along two-branch trajectories [10, 11].

2. Approximate solution of the problem in the Kármán's formulation. Let us consider the problem of finding the point of bifurcation of a compressed rod of cruciform transverse cross section (Fig. 2) in the Kármán's formulation $\delta P = 0$. The system of equations determining a nonrectilinear state of equilibrium of the rod has the form [12]

*) Editor's Note. The symbols for aluminium alloys appearing in this paper are given in the original Russian nomenclature.

$$\delta N = 0, \quad \frac{d^2}{dz^2}(\delta M_x) = Pp, \quad \frac{d^2}{dz^2}(\delta M_y) = Pq \tag{2.1}$$

$$\frac{d}{dz}(\delta M_z) = r^2 P \frac{d\theta}{dz} \quad \left(r^2 = \frac{1}{F}\int_F (x^2 + y^2)\, dF\right)$$

where the variations of the internal stresses are

$$\delta N = \int_F \delta\sigma_z dF, \quad \delta M_x = \int_F y\delta\sigma_z dF, \quad \delta M_y = -\int_F x\delta\sigma_z dF$$

$$\delta M_z = \int_F (x\delta\tau_{yz} - y\delta\tau_{xz})\, dF \tag{2.2}$$

while the variations of the curvature and the relative angle of torsion are, respectively,

$$p = -d^2 v \,/\, dz^2, \; q = d^2 u \,/\, dz^2, \; \theta = d\varphi \,/\, dz$$

Here u and v are the components of the translation of the points on the rod axis in the ox and oy direction, and φ is the angle of torsion.

Following the theory of thin-walled rods [12] with null sectorial characteristics, we set

$$\delta\varepsilon_z = \delta\varepsilon_0 - qx + py, \; \delta\gamma_{xz} = -2\,\theta y, \; \delta\gamma_{yz} = 2\,\theta x \tag{2.3}$$

where $\delta\varepsilon_z$, $\delta\gamma_{xz}$ and $\delta\gamma_{yz}$ are the variations of the axial and shear strain in the transverse cross section and $\delta\varepsilon_0$ is the variation of the deformation of the rod axis. The system (2.1) – (2.3) together with (1.1) forms a complete system of equations for studying the problem of bifurcation of the modes of equilibrium of a centrally compressed rod for $\delta P = 0$.

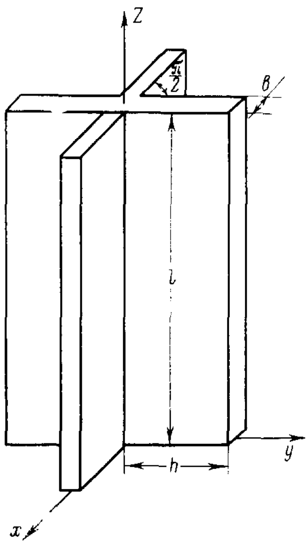

Fig. 2

We first consider an approximate solution of this problem. As we know, within the elastic limits a centrally compressed rod of cruciform transverse cross section may (depending on the geometrical dimensions) lose its stability under pure bending, or pure torsion. We shall assume for the time being that outside the elastic limits two modes of instability are also possible, the first one occurring when the torsional strain predominates the bending strain so that

$$\delta\varepsilon_z^{\max} \ll \delta\gamma^{\max} \; (\delta\gamma = (\delta\gamma_{xz}^2 + \delta\gamma_{zy}^2)^{1/2}),$$

and the second one taking place when the bending strain predominates the torsional strain $(\delta\gamma^{\max} \ll \delta\varepsilon_z^{\max})$. In both cases we neglect the quantity $\mu = \eta^2 \ln \eta^2$ which is small compared with unity. (Here η assumes the value of $\delta\varepsilon_z^{\max} / \delta\gamma^{\max}$ or of $\delta\gamma^{\max} / \delta\varepsilon_z^{\max}$ depending on which case is being considered).

Let us consider the first mode when $\eta = \delta\varepsilon_z^{\max} / \delta\gamma^{\max}$. Using the first and fourth equations of (2.1) we obtain, within the indicated accuracy,

$$\delta\varepsilon_0 = \frac{\sqrt{3}}{12 a_1}(a_0 - 2a_2)\, b \,|\, \theta \,| \tag{2.4}$$

$$k \frac{d}{dz}\left[\frac{1}{6} b_1 b^3 h\theta - \sqrt{3}\, b_2 b^2 h \delta\varepsilon_0 \frac{\theta}{|\theta|}\right] = Pr^2\theta' \tag{2.5}$$

The variations of p and q do not appear in the above equations. This suggests that the torsional mode of instability is not accompanied, as in the elastic region, by the bending, and we have $p = q = 0$. The present case differs from the elastic case in the fact that outside the limits of elasticity the twisting is accompanied, in accordance with (2.4), by additional compression of the rod axis. This result which is obtained from the relations (1.1) is mentioned neither by the deformation theory nor by the incremental theory of plasticity, although it is well confirmed by experiments [13]. Substituting (2.4) into (2.5) and taking into account the fact that $\theta' \not\equiv 0$, we obtain the final equation defining the critical stress σ_* in the Kármán's formulation

$$\frac{2\sigma_*}{k}\left[b_1 - \frac{3b_2}{4a_1}(a_0 - 2a_2)\right]^{-1} = \left(\frac{b}{h}\right)^2 \qquad (\delta_* = P_*/F) \tag{2.6}$$

Here $a_i = a_i(\sigma_*)$ and $b_i = b_i(\sigma_*)$ are known functions. For the particular case of the alloy AMΓ these functions are determined by the relations (1.2) and (1.3).

We note that the formula (2.6) has been obtained independently of the form of the boundary conditions, i.e. the critical load in the case of torsional mode of instability is determined, as in the elastic region, independently of the manner in which the rod is clamped.

In deriving (2.6) the quantity $\mu = \eta^2 \ln \eta^2$ $(\eta = \delta\varepsilon_z^{\max} / \delta\gamma^{\max})$ was assumed small compared with unity, and therefore it was neglected. We shall now obtain its estimate. In the case under consideration $p = q = 0$ and $\delta\varepsilon_z^{\max} = \delta\varepsilon_0$, hence using (2.4) we obtain

$$\eta = \frac{\sqrt{3}}{12a_1}(a_0 - 2a_2) \tag{2.7}$$

Numerical computations performed for the aluminium alloy AMΓ show that for $\sigma_* \leqslant 1.3\,\sigma_s$ the quantity $\mu \leqslant 0.07$.

Let us now consider another case, i.e. assume that $\eta = \delta\gamma^{\max} / \delta\varepsilon_z^{\max} \ll 1$ (later we shall show that $\eta = 0$). Neglecting the quantity $\mu = \eta^2 \ln \eta^2$ on assumption that it is small compared with unity, we obtain from the first three equations of (2.1), with (2.2) and (1.1) taken into account, that

$$4a_1 - \left(\frac{a_0}{2} + a_2\right)\left(\frac{\delta\varepsilon_0}{hq} + \frac{hq}{\delta\varepsilon_0}\right) + \left(\frac{a_0}{2} + a_2\right)\left(\frac{\delta\varepsilon_0}{hp} + \frac{hp}{\delta\varepsilon_0}\right) = 0 \tag{2.8}$$

$$\frac{3}{2} k \frac{d^2}{dz^2}\left[\frac{2}{3} a_1 b h^3 q - \left(\frac{a_0}{2} + a_2\right)\left(bh^2\delta\varepsilon_0 - \frac{b}{3}\frac{\delta\varepsilon_0^3}{q^2}\right)\right] = Pq \tag{2.9}$$

$$\frac{3}{2} k \frac{d^2}{dz^2}\left[\frac{2}{3} a_1 b h^3 p - \left(\frac{a_0}{2} + a_2\right)\left(bh^2\delta\varepsilon_0 - \frac{b}{3}\frac{\delta\varepsilon_0^3}{p^2}\right)\right] = Pp$$

We see that θ does not appear in the above equations, and the equations determine the pure bending mode of instability of a centrally compressed rod. Consequently $\eta = \mu = 0$. In this case already known formulas can be used to find σ_*.

In particular, when both ends of the rod are hinged, we have

$$\sigma_* = 4\pi^2 E E_t / \lambda^2 (\sqrt{E} + \sqrt{E_t})^2, \qquad \lambda = l\sqrt{F / I_{\min}}$$

where λ denotes the flexibility of the rod and $E_t = d\sigma / d\varepsilon$ is the tangential modulus defined from the stress-strain diagram $\sigma \sim \varepsilon$ for uniaxial tension or compression. For the practical determination of the critical load and explanation of the possible mode of instability, one must use the expressions (2.6) and (2.9) and choose the smaller of the two values of σ_* obtained.

3. Exact solution of the problem of torsional mode of instability. Comparison of the results obtained with those already known in literature. In Sect. 2 it was shown that the torsional mode of instability is accompanied by an additional shortening of the rod axis, without however any bending, i.e. $p = q = 0$. We shall use this result to construct an exact solution for the problem on the torsional mode of instability, without restricting the quantity μ.

The first and fourth equation of (2.1), relations (1.1), (2.2), (2.3) and the condition $p = q = 0$, yield

$$4a_1\delta\varepsilon_0 + (2a_2 - a_0)\,\delta\varepsilon_u^* - \frac{\sqrt{3}}{2}(6a_2 + a_0)\frac{\delta\varepsilon_0^2}{b\theta}\ln\frac{\sqrt{3}\,\delta\varepsilon_u^* + b\theta}{\sqrt{3}\,\delta\varepsilon_u^* - b\theta} = 0$$

$$\frac{1}{2}k\frac{d}{dz}\left[\frac{4}{3}hb^3b_1\theta - 6hb^2b_2\frac{\delta\varepsilon_0\delta\varepsilon_u^*}{b\theta} + 3\sqrt{3}\,hb^2b_2\frac{\delta\varepsilon_0^3}{b^2\theta^2}\ln\frac{\sqrt{3}\,\delta\varepsilon_u^* + b\theta}{\sqrt{3}\,\delta\varepsilon_u^* - b\theta}\right] = Pr^2\theta' \qquad (3.1)$$

$$\left(\delta\varepsilon_u^* = \sqrt{\delta\varepsilon_0^2 + (b\theta/\sqrt{3})^2}\right)$$

Setting now $\theta \not\equiv 0$ and $\theta' \not\equiv 0$ and denoting $\delta\varepsilon_0/b\theta = s$ we obtain, from the second relation of (3.1), the following equation for the critical stress σ_*:

$$\sigma^* = \frac{1}{2}k\left[b_1 - 3\sqrt{3}\,b_2 s\sqrt{3s^2+1} + \frac{9\sqrt{3}}{2}b_2 s^3\ln\frac{\sqrt{3s^2+1}+1}{\sqrt{3s^2+1}-1}\right]\left(\frac{b}{h}\right)^2 \qquad (3.2)$$

The quantity s can be found from the first equation of (3.1) which can be rewritten as

$$8\sqrt{3}a_1 s + 2(2a_2 - a_0)\sqrt{3s^2+1} - 3(6a_2 + a_0)s^2\ln\frac{\sqrt{3s^2+1}+1}{\sqrt{3s^2+1}-1} = 0 \qquad (3.3)$$

The actual computations of the relation $(\sigma_*/\sigma_s) \sim (h/b)$ were performed for the aluminium alloy АМГ on a computer according to the following scheme. For the given σ_* the value of s was determined from (3.3) using the Newton's method and then the ratio h/b was found from (3.2).

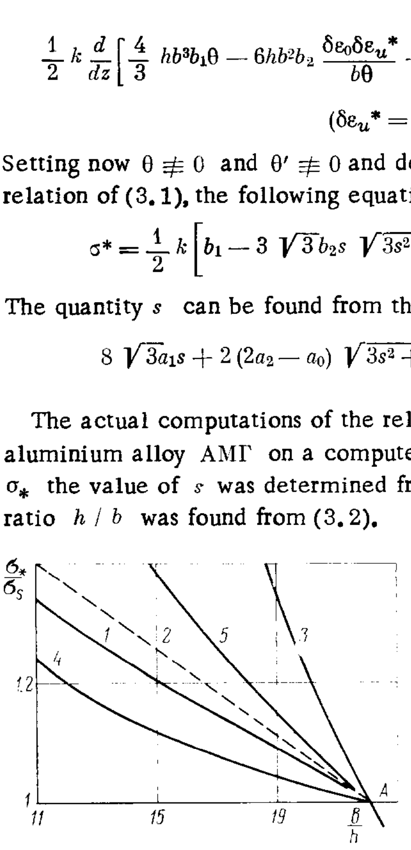

Fig. 3

The relation $(\sigma_*/\sigma_s) \sim (h/b)$ given by (3.2) and (3.3) is depicted on Fig. 3 by the solid line 1. The dashed line 2 corresponds to the approximate formula (2.3). Line 3 depicts the formula $\sigma_* = (b/h)^2 G$ obtained in [3] within the framework of the incremental theory. Line 4 is based on the deformation theory which leads to the expression $\sigma_* = (b/h)^2 G_s$, where $G_s = \tau_{oct}/\gamma_{oct}$ is the secant modulus on the octahedral shear versus octahedral stress diagram. The point A corresponds to the transition from the elastic to the plastic region.

Line 5 in Fig. 3 helps to determine the effect of additional compression $\delta\varepsilon_0$ of the rod axis on the quantity σ_* in the torsional mode of instability. It is constructed from Eqs. (3.2) and (3.3) for $s = 0$ $(s = \delta\varepsilon_0/b\theta)$.

We note that the corresponding plots for the aluminium alloys АК-6 and Д-16 are qualitatively identical to those given in Fig, 3 for the alloy АМГ.

4. Solution of the problem in the Shanley's formulation. The solution in the Kármán's formulation ($\delta P = 0$) obtained above shows that neither the deformation theory nor the incremental theory yield satisfactory results when the rod is in the torsional mode of instability with well developed plastic deformation. First let us deal with the problem whether it is possible to use the deformation theory to solve the problem in the Shanley's formulation, i.e. when $\delta P > 0$. We assume that increase in the external load leads to the angles β of the break in the load trajectory (Fig. 1) being smaller than the limiting angle β_0 at which the deformation theory may apply, at all points of the transverse cross section of the rod, i.e. $\beta \leqslant \beta_0\ (\sigma_0)$ $(\beta_0 < \pi / 2)$. Here we can write the following expressions for the variations in stress

$$\delta\sigma_z = E_t\delta\varepsilon_z, \qquad \delta\tau_{xz} = G_s\delta\gamma_{xz}, \qquad \delta\tau_{yz} = G_s\delta\gamma_{yz} \tag{4.1}$$

Next we replace the first and the fourth equation of (2.1) by the following expressions (here we make use not only of the physical, but also of the geometrical nonlinearity)

$$\delta N = \delta P, \qquad \frac{d}{dz}(\delta M_z) = P\theta'\Phi(\theta) \qquad \left(\Phi(\theta) = \int_F \frac{(x^2+y^2)\,dE}{[1+(x^2+y^2)\,\theta^2]^{3/2}}\right) \tag{4.2}$$

where δN and δM_z are determined by the quadratures (2.2). Using (4.2) and taking into account (4.1), we obtain

$$E_t\delta\varepsilon_0 = \delta\sigma, \ \sigma = G_s I_d / \Phi(\theta), \ \sigma = P / F \tag{4.3}$$
$$\delta\sigma = \delta P / F, \ I_d = {}^4/_3\, hb^3$$

The second formula of (4.3) yields the following expression for the ratio $\delta\sigma / \delta\tau = \delta\sigma / (G_s\delta\gamma)$ near the break in the trajectory (Fig. 1)

$$\frac{\delta\sigma}{\delta\tau} = \frac{\sigma\theta}{G_s}\left[\frac{\delta G_s}{\delta\sigma} I_d - \Phi(\theta)\right]^{-1}\Phi(\theta) \tag{4.4}$$

On passing to the limit as $\theta \to 0$ we obtain $\lim \delta\sigma / \delta\tau = 0$, i.e. the angle $\beta = \mathrm{arctg}\,(\delta\tau / \delta\sigma) = \pi / 2$. But this contradicts our previous assumption that the angles β are bounded by the limiting angle $\beta_0 < \pi / 2$ which defines the limits of applicability of the deformation theory. Consequently in the Shanley's formulation the nonlinear differential relations (1.1) must also be used in solving the problem. Repeating the computations made in Sect. 2, we arrive at the following conclusion. In place of (2.4) we have

$$\delta\varepsilon_0 = \frac{\delta P}{kF} - \frac{\sqrt{3}}{12a_1}(a_0 - 2a_1)\,b\theta \tag{4.5}$$

Equation (2.5) remains unchanged. Noting that $d(\delta P) / dz = 0$, we find that in the Shanley's case the formula for the critical load obtained from (2.5) and (4.5) is identical to (2.6) which was obtained earlier for the Kármán's case. Thus in the case of the torsional mode of instability of a compressed rod of cruciform transverse cross section the critical loads coincide in both the Karman's and the Shanley's formulation.

If instability occurs in the bending mode, then (2.9) must be replaced by $\sigma_* = \pi^2 E_t / \lambda^2$ when the problem is being solved in the Shanley's formulation.

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